

Micah

Homotopy Hypothesis: there should be a theory of "n-categories" where the n-groupoids are models of "n-types"

n-type: a space X where $\pi_k X = 0$ for $k > n$

Ex if X is a 1-type, then $\pi_1 X$ is a 1-groupoid
 \downarrow
fundamental groupoid

Taking $n = \infty$, ∞ -groupoids should be homotopy types

If we want to study $(\infty, 1)$ -categories, this suggests that we should be able to model these by categories enriched in "spaces" (homotopy types)

If we model ∞ -cats by quasi-categories, then this "idea" holds

(Janis + Micah fight)

Ex Top category of compactly generated weak Hausdorff spaces

$\text{Map}_{\text{Top}}(X, Y)$ as the set of continuous maps with the compact-open topology

$$\text{Top} : X \begin{array}{c} \xleftarrow{|-|} \\ \xrightarrow{\text{Sing}(-)} \end{array} X_0 : \mathcal{S}\text{Set}$$

is a Quillen adjunction.

So, categories enriched in "spaces" should be the same as categories enriched in $\mathcal{S}\text{Set}$

$$\text{Map}(X, Y) \times \text{Map}(Y, Z) \longrightarrow \text{Map}(X, Z)$$

$$|\text{Map}(X, Y) \times \text{Map}(Y, Z)| \longrightarrow |\text{Map}(X, Z)|$$

$$|\text{Map}(X, Y)| \times |\text{Map}(X, Z)| \xrightarrow{\quad} |\text{Map}(X, Z)|$$

Cat_Δ is the category of simplicially-enriched categories, Cat_{Top} is topologically-enriched categories

So geometric realization induces a functor $\text{Cat}_\Delta \longrightarrow \text{Cat}_{\text{Top}}$

$\text{Sing}(-)$ also commutes with limits, so the same property holds

$\mathcal{C} \in \text{Cat}_\Delta$, $N(\mathcal{C})_0 = \text{objects}$

$N(\mathcal{C})_1 = \text{Hom}_{\mathcal{C}}(-, -)$

a 2-simplex in $N(\mathcal{C})$ is a diagram

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z \end{array}$$

but in this case $h = g \circ f$; but we just want $h \simeq g \circ f$

Recall $N(\mathcal{C}) := \text{Hom}_{\text{Cat}}([2], \mathcal{C})$ (i.e. $\text{Fun}([2], \mathcal{C})$)

$$[2] \quad \begin{array}{c} 0 \\ \bullet \end{array} \xrightarrow{1} \begin{array}{c} 1 \\ \bullet \end{array} \xrightarrow{2} \begin{array}{c} 2 \\ \bullet \end{array}$$

$$\varphi_{02} = \varphi_{12} \circ \varphi_{01}$$

Idea: freely add elements to hom sets of $[n]$ s.t. $\varphi_{ij} \neq \varphi_{jk} \circ \varphi_{ij}$

$$S[2] \quad \begin{array}{c} 0 \\ \bullet \end{array} \xrightarrow{\varphi_{01}} \begin{array}{c} 1 \\ \bullet \end{array} \xrightarrow{\varphi_{12}} \begin{array}{c} 2 \\ \bullet \end{array} \quad \varphi_{02} \neq \varphi_{01} \circ \varphi_{12}$$

$\underbrace{\hspace{10em}}_{\varphi_{02}}$

Define $S[n]$ to be this "thickened up" version of $[n]$

So now a functor $S[2] \rightarrow \mathcal{C}$ is the data of

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z \end{array} \quad \text{and of a homotopy } h \simeq g \circ f$$

This is a functor $\Delta \rightarrow \text{Cat}_s$ which sends $[n]$ to $S[n]$.

So define a simplicial set $N(\mathcal{C})$

$$\text{Hom}_{\text{sSet}}([n], N(\mathcal{C})) \cong \text{Hom}_{\text{Cat}_\Delta}(S[n], \mathcal{C})$$

Rmk $\text{sSet} := \text{Fun}(\Delta^{\text{op}}, \text{Set})$

the Yoneda embedding realizes this as the "colimit completion" of Δ

So functors out of Δ into a category \mathcal{C} with colimits are equivalent to colimit-preserving functors $\text{sSet} \rightarrow \mathcal{C}$.

So, our functor $\Delta \rightarrow \text{Cat}_\Delta$ lifts to a colimit-preserving functor $\text{sSet} \rightarrow \text{Cat}_\Delta$

\Rightarrow Adjoint functor theorem implies that there exists a functor $\text{Cat}_\Delta \rightarrow \text{sSet}$ right adjoint to the first

$$S: \text{sSet} \rightleftarrows \text{Cat}_\Delta : N$$

To recap: To any simplicial cat $\mathcal{C} \in \text{Cat}_\Delta$, we obtain a sSet , $N(\mathcal{C})$.
Then $N(\mathcal{C})$ is an ∞ -cat

Then these functors N and S are equivalences

def The topological nerve

$$\text{Cat}_{\text{Top}} \rightarrow \text{sSet}$$

is defined by $N \circ \text{Sing}(-)$

Ex Top is now an ∞ -cat by associating it with its topological nerve

Mapping Spaces in ∞ -categories

Let \mathcal{C} be an ∞ -cat
for two objects $x, y \in \mathcal{C}$,

$$\text{Map}_{\mathcal{C}}(x, y) := \text{Map}_{\text{hs}}(x, y)$$

i.e. $S(\mathcal{C})(x, y) \in \mathcal{K} \rightarrow$ simplicial sets localized at weak equivalences

No obvious maps

$$\text{Map}_{\text{hs}}(x, y) \times \text{Map}_{\text{hs}}(y, z) \rightarrow \text{Map}_{\text{hs}}(x, z)$$

Jānis

$S: \Delta \rightarrow \text{Cat}_\Delta$ extends to $S: s\text{Set} \rightarrow \text{Cat}_\Delta$
with right adjoint $N: \text{Cat}_\Delta \rightarrow s\text{Set}$

$\text{Ho}: s\text{Set} \rightleftarrows \text{Cat}_1: N$ (not simplicial nerve)

See HTT 1.2.3, Cisinski 1.4-1.6 "Boardman-Vogt constr."

Let \mathcal{C} be an ∞ -cat.

Goal: Construct a 1-cat $\text{Ho}(\mathcal{C})$.

Construct $\pi(\mathcal{C}) \cong \text{Ho}(\mathcal{C})$

What is a 1-cat?

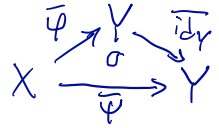
- objects
- morphisms
- composition
- identity morphisms
- associativity of comp.

$\text{Ob}(\pi(\mathcal{C})) = \mathcal{C}_0$

Morphisms: For every $(\varphi: \Delta^1 \rightarrow \mathcal{C}) \in \mathcal{C}_1$,
put $\bar{\varphi} \in \text{Edge}(\varphi(0), \varphi(1))$

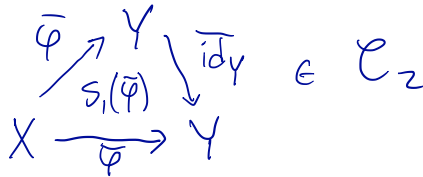
We also have $\text{id}_X \in \text{Edge}(X, X)$ (degen. 1-simplex)

def let $\bar{\varphi}, \bar{\psi} \in \text{Edge}(X, Y)$. Then $\bar{\varphi}$ and $\bar{\psi}$ are homotopic if \exists a 2-cell $\sigma \in \mathcal{C}_2$ s.t. $\partial_2(\sigma) = \bar{\varphi} - \bar{\psi}$
 $\partial_1(\sigma) = \bar{\varphi}$
 $\partial_0(\sigma) = \text{id}_Y$



Prop For every $\varphi \in \mathcal{C}_1$ with $\varphi(0) = X$ and $\varphi(1) = Y$, relation on $\text{Edge}(X, Y)$ is an equivalence

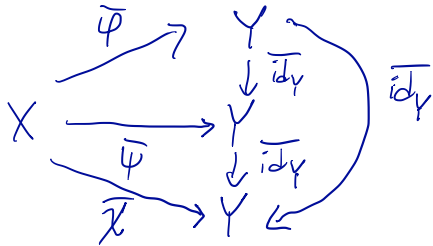
Pf Reflexivity:



Symmetry/trans:

$\bar{\varphi}, \bar{\psi}, \bar{\chi} \in \text{Edge}(X, Y)$

assume $\bar{\varphi} \sim \bar{\psi}$
 $\bar{\psi} \sim \bar{\chi}$



this is $\Lambda_2^3 \rightarrow \mathcal{C}$, lifts to $\tau: \Delta^3 \rightarrow \mathcal{C}$

Then $d_2(\tau)$ shows that $\bar{\varphi} \sim \bar{\chi}$

for symmetry, take same diagram going other direction.

def for $x, y \in \text{Ob}(\pi(\mathcal{C})) = \mathcal{C}_0$, let

$$\text{Hom}_{\pi(\mathcal{C})}(x, y) = \text{Edge}(x, y) / \text{homotopy}$$

$$(\varphi: \Delta^1 \rightarrow \mathcal{C}) \in \mathcal{C}_1 \rightsquigarrow \bar{\varphi} \in \text{Edge}(x, y) \rightsquigarrow [\bar{\varphi}] \in \text{Hom}_{\pi(\mathcal{C})}(\varphi(0), \varphi(1))$$

Composition: $[\bar{\varphi}] \in \text{Hom}_{\pi(\mathcal{C})}(x, y)$, $[\bar{\psi}] \in \text{Hom}_{\pi(\mathcal{C})}(y, z)$

$$\begin{array}{ccc} \bar{\varphi} & \rightarrow & Y \\ X & \nearrow & \searrow \bar{\psi} \\ & \Lambda^2 & \rightarrow & Z \\ & \Lambda^2 & \rightarrow & \mathcal{C} \end{array} \quad \text{extends to } \sigma: \Delta^2 \rightarrow \mathcal{C}$$

$$[\bar{\psi}] \circ [\bar{\varphi}] = [d_1 \sigma]$$

Is comp. well defined?

① Choice of σ is ok:

$$\begin{array}{ccc} \Lambda^3 & \rightarrow & \mathcal{C} \\ \tau: \Delta^3 & \rightarrow & \mathcal{C} \end{array}$$

$$\begin{array}{ccccc} & & & \xrightarrow{\bar{\psi}} & Z \\ X & \xrightarrow{\bar{\varphi}} & Y & \xrightarrow{\bar{\psi} \circ \bar{\varphi}} & Z \\ & \searrow & & \downarrow \text{id}_Z & \\ & & & & Z \end{array}$$

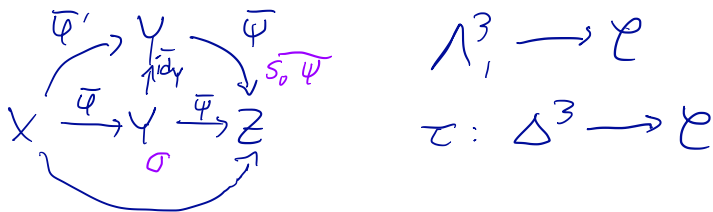
$$d_2 d_1 \tau = d_1 \sigma$$

$$d_1 d_1 \tau = d_1 \sigma'$$

$$d_0 d_1 \tau = \text{id}_Z$$

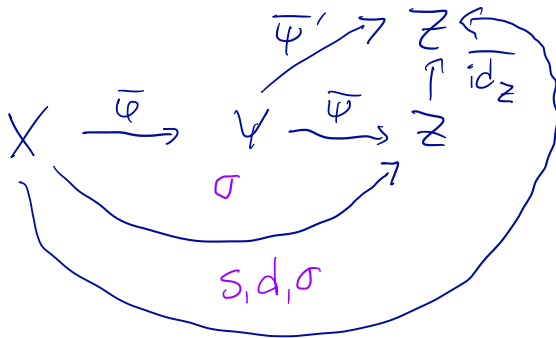
Now $d_1 \tau$ shows that $d_1 \sigma \sim d_1 \sigma'$

② Choice of $\bar{\varphi}$ is ok:



So $d_1 \tau$ says: $[\bar{\psi}] \circ [\bar{\varphi}'] = \overline{[d_1 d_1 \tau]}$
 $\approx [d_1 \sigma]$
 $= [\bar{\psi}] \circ [\bar{\varphi}].$

③ Choice of $\bar{\psi}$ is ok:

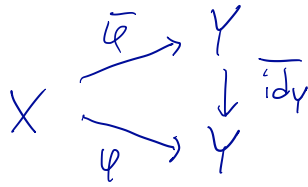


$\Lambda^3_2 \rightarrow \mathcal{C} \Rightarrow \tau: \Delta^3 \rightarrow \mathcal{C}$

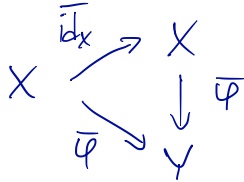
Here $d_2 \tau$ shows that

$[\bar{\psi}'] \circ [\bar{\varphi}] = [\bar{\psi}] \circ [\bar{\varphi}].$

Identity:

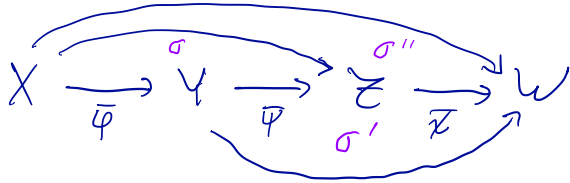


$\overline{S_1 \varphi}$ says $[\bar{id}_Y] \circ [\bar{\varphi}] = [\bar{\varphi}]$



$\overline{S_0 \varphi}$ says $[\bar{\varphi}] \circ [\bar{id}_X] = [\bar{\varphi}]$.

Associativity:



$\Lambda_2^3 \rightarrow \mathcal{C} \Rightarrow \tau: \Delta^3 \rightarrow \mathcal{C}$

$$\begin{aligned}
 [\bar{\pi}] \circ ([\bar{\varphi}] \circ [\bar{\varphi}]) &= [\overline{d_1 \sigma''}] \\
 &= [\overline{d_1 d_2 \tau}] \\
 &= [d_1 \sigma'] \circ [\bar{\varphi}'] \\
 &= ([\bar{\pi}] \circ [\bar{\varphi}]) \circ [\bar{\varphi}]
 \end{aligned}$$

Therefore, $\pi(\mathcal{C})$ is a 1-cat.

□

Ex

- View a $\mathcal{C} \in \text{Cat}_1$ as $\mathcal{C} \in \text{Cat}_\infty$. Then $\text{Ho}(\mathcal{C}) = \mathcal{C}$, since $\mathcal{C}_{12} = \mathcal{C}_1$.
- Let X be a top. space. Then $\text{Ho}(\text{Sing}(X)) = \pi_1 X$.
- Let $\mathcal{C} \in \text{Cat}_\Delta$, then $N(\mathcal{C}) \in \text{Cat}_\infty$.
If $\text{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex,
then $\text{Ho}(N(\mathcal{C})) = h\mathcal{C}$.

Maximilien

limits and colimits in ∞ -categories

Goal: introduce the notion formally then present an equivalent approach with homotopy (co)limits via topological (or simplicial) enrichments

def Let \mathcal{C} be an ∞ -category. Let $p: K \rightarrow \mathcal{C}$ be a map of simplicial sets. A limit of p is a final object in \mathcal{C}_p . A colimit of p is an initial object in \mathcal{C}_p .

① Quick reminder for ordinary categories

\mathcal{C} an ordinary cat, $X \in \text{Ob}(\mathcal{C})$.

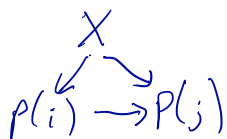
over category \mathcal{C}/X : $\text{Ob}: \begin{array}{c} X' \\ \downarrow \\ X \end{array} \in \mathcal{C}$

$\text{Mor}: \begin{array}{ccc} X' & X'' & X' \rightarrow X'' \\ \downarrow & \downarrow & \downarrow \downarrow \\ X & X & X \end{array} = \begin{array}{ccc} X' & & X'' \\ & \downarrow & \downarrow \\ & X & X \end{array}$

Can be generalized: $X: [0] \rightarrow \mathcal{C}$
 \hookrightarrow cat w/ 1 obj + id

Given a functor $p: I \rightarrow \mathcal{C}$, denote by \mathcal{C}_p the slice category of cones of p .

\mathcal{C}_p are objects of the form



Dually we have notion of cocones \mathcal{C}_p .

Recall: $\lim(p) \simeq$ terminal object of \mathcal{C}_p .

② Joins of ∞ -cats


Given A, B ordinary cats, define

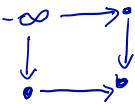
$$A \star B: \quad \text{ob}(A \star B) = \text{ob}(A) \amalg \text{ob}(B)$$

$$\text{Hom}_{A \star B}(X, Y) \begin{cases} \text{Hom}_A(X, Y), & X, Y \in A \\ \text{Hom}_B(X, Y), & X, Y \in B \\ \star, & X \in A, Y \in B \\ \emptyset, & X \in B, Y \in A. \end{cases}$$

def $[0] \star B =: B^{\blacktriangleright}$, the category of cones of B

$B \star [0] =: B^{\blacktriangleleft}$, the category of cocones of B

ex $B =$ 

$B^\nabla =$ 

This construction can be extended to simplicial sets such that

$$N(A) * N(B) \xrightarrow{\cong} N(A * B)$$

(regular nerve)

def $K, L \in \mathbf{sSet}$. $(K * L)_m = K_m \cup L_m \cup \bigcup_{i+j=m-1} K_i \times L_j$

ex $\Delta^0 * \Delta^0 \cong \Delta^1$

In general, $\Delta^n = \underbrace{\Delta^0 * \dots * \Delta^0}_{n+1 \text{ times}}$

Even more general, $\Delta^i * \Delta^j \cong \Delta^{i+j}$

\exists Canonical inclusions $K \hookrightarrow K * L, L \hookrightarrow K * L$

$\rightsquigarrow \forall L \in \mathbf{sSet}, \mathbf{sSet} \rightarrow (\mathbf{sSet})_L$

$K \rightarrow K * L,$

this functor preserves colimits

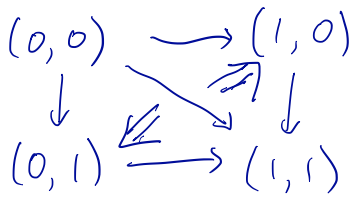
def $\Delta^\circ \star L =: L^\Delta$ cone of L

$K \star \Delta^\circ =: K^\triangleright$ cocone of K

ex $\Lambda_2^z =$

$$C_0 \longrightarrow C_2 \xleftarrow{c_1} C_1$$

$(\Lambda_2^z)^\Delta \cong \Delta' \times \Delta'$



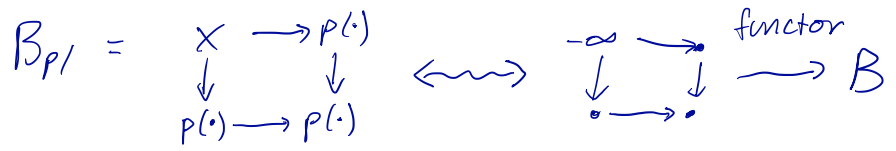
③ Sliced ∞ -categories

In ordinary categories, given a functor $p: A \rightarrow B$, notice $B_{p/}$ is characterized by

$\text{Fun}(\mathcal{C}, B_{p/}) \cong \text{Fun}_p(\mathcal{C} \star A, B)$



$\mathcal{C} = [0] \quad A = \dots \rightarrow !$



Worthwhile rewriting the iso:

$$\text{Mor}_{\text{cat}}(\mathcal{C}, \mathcal{B}/p) \cong \text{Hom}_{\text{Cat}_{A_1}} \left(\begin{array}{c} A \\ \downarrow \\ C \star A \end{array}, \begin{array}{c} A \\ \downarrow p \\ B \end{array} \right)$$

Prop Let $p: \mathcal{L} \rightarrow \mathcal{C}$ a map of $s\text{Set}$ where \mathcal{C} is an ∞ -cat. Then there is an ∞ -cat \mathcal{C}_p characterized by

$$\forall K \in s\text{Set}: \text{Hom}_{s\text{Set}}(K, \mathcal{C}_p) \cong \text{Hom}_{s\text{Set}_{\mathcal{L}}} \left(\begin{array}{c} \mathcal{L} \\ \downarrow \\ K \star \mathcal{L} \end{array}, \begin{array}{c} \mathcal{L} \\ \downarrow p \\ \mathcal{C} \end{array} \right)$$

$$(\mathcal{C}_p)_n = \text{Hom}_{s\text{Set}_{\mathcal{L}}} \left(\begin{array}{c} \mathcal{L} \\ \downarrow \\ \Delta^n \star \mathcal{L} \end{array}, \begin{array}{c} \mathcal{L} \\ \downarrow p \\ \mathcal{C} \end{array} \right)$$

(Dually, we have \mathcal{C}_p where replace $K \star \mathcal{L}$ with $\mathcal{L} \star K$)

④ initial objects/final objects

An obj $X \in \mathcal{C}$ is final if in its homotopy $\text{Ho}(\mathcal{C})$, $\forall Y \in \mathcal{C}$, $\text{Mor}_{\text{Ho}(\mathcal{C})}(X, Y) \sim \ast$

\mathcal{C} can be top cat, simplicial cat, or ∞ -cat.

⑤ ∞ -(co)limits as homotopy (co)-limits

Unpacking the definition: $\lim(K \xrightarrow{p} \mathcal{C})$ is

the final object of \mathcal{C}_p . An object of \mathcal{C}_p is a vertex of $(\mathcal{C}_p)_0$.

Recall: $(\mathcal{C}_p)_0 = \text{Hom}_{\text{Set}_K} \left(\begin{array}{c} K \\ \Delta^0 \downarrow \\ \Delta^0 * K \end{array}, \begin{array}{c} K \\ \mathcal{C} \downarrow p \end{array} \right)$

i.e. a functor $\bar{p}: K^\Delta = \Delta^0 * K \rightarrow \mathcal{C}$ such that

$$\bar{p}|_K = p$$

Often we denote $\bar{p}(-\infty) \in \mathcal{C}$ as $\lim(p)$
 \rightsquigarrow the usual nerve preserves (co)limits

Recall Given an ∞ -cat \mathcal{C} , it is enriched topologically (or simplicially):

given objects $X, Y \in \mathcal{C}$, $\text{Map}_{\mathcal{C}}(X, Y)$

can be regarded as the object in the homotopy category of spaces representing the space of maps from X to Y in $\text{Ho}(\mathcal{C})$

Recall 2: For usual limits in ordinary cats:

let $L = \lim (I \xrightarrow{p} \mathcal{C})$, then

$$\forall X \in \mathcal{C}: \text{Map}_{\mathcal{C}}(X, L) \xrightarrow{\cong} \text{Mor}_{\mathcal{C}^I}(\underline{X}, p)$$

\uparrow
 const. diag.

Prop Let \mathcal{C} be an ∞ -cat and K a sSet, $p: K \rightarrow \mathcal{C}$ a sSet map. Then

$\bar{p}: K^{\Delta} \rightarrow \mathcal{C}$ is a limit of $p = \bar{p}|_K$

\iff if we denote $L = \bar{p}(-\infty)$, $\forall X \in \mathcal{C}$,

$$\text{Map}_{\mathcal{C}}(X, L) \xrightarrow{\sim} \text{Map}_{\text{Fun}(K, \mathcal{C})}(\underline{X}, p)$$

Ex Recall that homotopy product in Top is given by the product in Ho(Top)

If \mathcal{C} is a topological cat, then homotopy product $\prod_{\alpha} Y_{\alpha} \in \mathcal{C}$ is determined

$$\forall X \in \mathcal{C}: \text{Map}(X; \prod_{\alpha} Y_{\alpha}) \xrightarrow{\sim} \prod_{\alpha} \text{Map}(X, Y_{\alpha})$$

\uparrow
 weak

ex recall that the homotopy pullback in Top :

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is given by the space

$$\{(x, y, h) \in X \times Y \times \text{Map}([0, 1], Z) \mid h(0) = f(x), h(1) = g(y)\} \\ \subseteq X \times Y \times \text{Map}([0, 1], Z)$$

For a topological cat \mathcal{C} , P is a htpy pull back of $X \rightarrow Z$ in \mathcal{C} if:

$$\forall W: \text{Map}_{\mathcal{C}}(W, P) \xrightarrow{\sim} \text{Map}(W, X) \times_{\text{Map}(W, Z)}^n \text{Map}(W, Y)$$

def Let \mathcal{C} be a topological category. Define the topological nerve of \mathcal{C} :

$$N_{\text{Top}}(\mathcal{C}) = N_{\Delta}(\text{Sing}(\mathcal{C}))$$

def Let S be the ω -cat of spaces defined as $N_{\Delta}(\text{Kan})$ or $N_{\text{Top}}(\text{CW})$

Thm Let \mathcal{I}, \mathcal{C} be topological categories,

Let $p: \mathcal{I} \rightarrow \mathcal{C}$ be a continuous functor. Suppose we have a cone

$(C, \{\eta_i: C \rightarrow p(i)\}_{i \in \mathcal{I}})$, $(C, \{\eta_i\})$ is

a homotopy limit of $p \iff$

$p = N_{\text{Top}}(p): N_{\text{Top}}(\mathcal{I}) \rightarrow N_{\text{Top}}(\mathcal{C})$ can

be extended to a functor

$\bar{p}: N_{\text{Top}}(\mathcal{I})^{\infty} \rightarrow N_{\text{Top}}(\mathcal{C})$ is an ∞ -limit
in $N_{\text{Top}}(\mathcal{C})$.

∞ -limit in \mathcal{S} = homotopy limits in spaces

\uparrow
 \leftarrow

∞ -limits in any ∞ -cat

topological enrichments

\exists statement for simplicial as well

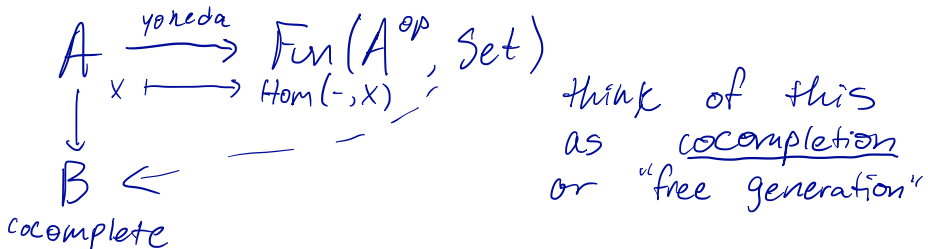
Harry

Presentable (ordinary) categories

- bridge the gap between small categories and proper classes
- presentable categories are "generated" by small categories

Ex Ab is a proper class, but Ab^{fg} is small

Recall for any small A , we have cocomplete presheaf category $Fun(A^{op}, Set)$



Thm For A small and C cocomplete, the restriction to A along the Yoneda embedding gives an equivalence

$$Fun_{\uparrow}^L(Fun(A^{op}, Set), C) \xrightarrow{\sim} Fun(A, C)$$

colim preserving functors

Cor $A = *$, $\text{Fun}^L(\text{Set}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}$

Thm (Adjoint Functor Thm) (Freyd)

For $F: \mathcal{C} \rightarrow \mathcal{D}$, \mathcal{C}, \mathcal{D} cocomplete, F is a left adjoint if and only if F preserves colimits and the "solution set condition" is satisfied

SSC: (roughly) some class of morphisms is a set

Presentable categories form a suitably large class of categories for which we can drop SSC.

def Let κ be a regular cardinal. Then a κ -filtered colimit is given by a filtered system where objects + morphisms are less than κ .

A category \mathcal{C} is κ -accessible if it contains some small subcat $\mathcal{D} \subseteq \mathcal{C}$ such that:

- every $X \in \mathcal{C}$ is a κ -filtered colim of objects in \mathcal{D} ($\mathcal{D} \rightarrow \mathcal{C}$ is dense)
- for $d \in \mathcal{D}$, $\text{Hom}(d, -)$ commutes with κ -filtered colimits (d 's are κ -compact in \mathcal{C})

def \mathcal{C} is accessible if it is κ -accessible for some κ and $F: \mathcal{C} \rightarrow \mathcal{D}$ is accessible if it preserves filtered colimits

def A category \mathcal{C} is presentable if it is accessible + cocomplete

Note: Any presentable cat is also complete

Ex • Set , as generated by $*$

- Any presheaf category $\text{Fun}(A^{\text{op}}, \text{Set})$ on a small category
- $s\text{Set} = \text{Fun}(\Delta^{\text{op}}, \text{Set})$
- Mod_R generated by f.g. projectives
- $\text{Ch}(R)$ generated by perfect complexes
- Quasicoherent \mathcal{O}_X -modules on any scheme

Classification for presentable categories

A category \mathcal{C} is presentable if and only if it is a localization of a presheaf category on a small accessible category

$$\begin{array}{ccc}
 A & \xrightarrow{\text{venda}} & \text{Fun}(A^{\text{op}}, \text{Set}) & \xrightarrow{\text{localization}} & \mathcal{C} \\
 & \uparrow & & & \uparrow \\
 & \text{free generation} & & & \text{relations}
 \end{array}$$

Adjoint Functor Thm for Presentable Categories:

$F: \mathcal{C} \rightarrow \mathcal{D}$, \mathcal{C}, \mathcal{D} presentable

① F is a left adjoint if and only if it preserves colimits

② F is a right adjoint if and only if it preserves limits and is accessible.

Presentable ∞ -categories

Recall ∞ -category of spaces $\mathcal{S} = N_{\Delta}(\text{kan})$

for some small simplicial set K ,
define ∞ -cat of presheaves

$$\mathcal{D}(K) := \text{Fun}(K^{\text{op}}, \mathcal{S})$$

Recall We have an adjunction

$$\begin{array}{ccc}
 (C[-], N_{\Delta}) : s\text{Set} & \rightleftarrows & \text{Cat}_{\Delta} \\
 \swarrow & & \searrow \\
 \text{simplicial thickening} & & \text{coherent nerve}
 \end{array}$$

For some simplicial set, take $C[K]$ and $C[K^{op}]$

$$C[K] \times C[K^{op}] \longrightarrow \text{Kan}$$

$$(X, Y) \longmapsto \text{Sing}(\text{Hom}_{C[K]}(X, Y))$$

We compose

$$C[K \times K^{op}] \longrightarrow C[K] \times C[K^{op}] \longrightarrow \text{Kan}$$

pass to adjoint

$$K \times K^{op} \longrightarrow N_{\Delta}(\text{Kan}) \simeq S$$

exponential law

$$K \longrightarrow \text{Fun}(K^{op}, S) \quad \text{this is } \infty\text{-Yoneda embedding}$$

- Fully faithful and satisfies necessary universal property

$$\underline{\text{Thm}} \quad \text{Fun}^L(\mathcal{D}(\mathcal{D}), \mathcal{C}) \xrightarrow{\sim} \text{Fun}(\mathcal{D}, \mathcal{C})$$

$$\underline{\text{Cor}} \quad \text{Take } \mathcal{D} = \Delta^0, \text{ get } \text{Fun}^L(S, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}$$

"The ∞ -cat of spaces is freely generated by the zero simplex Δ^0 "

def An ∞ -cat is presentable if it is cocomplete and accessible.

A functor between ∞ -cats is a localization if it has a fully faithful right adjoint

Classification of presentable ∞ -cats

∞ -cat \mathcal{C} is presentable if and only if it is an accessible localization of $\mathcal{P}(\mathcal{D})$ for some small ∞ -cat \mathcal{D} .

Towards understanding accessible localizations of ∞ -categories:

$(L, R) : \mathcal{C} \rightleftarrows \mathcal{D}$, denote $L : \mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{C}$ as well

let $S_L =$ the class of maps in \mathcal{C} sent to equivalences by L

Localizations are completely determined by S_L

In this case

$S_L \subseteq \text{Fun}([1], \mathcal{C})'$:

- closed under colimits
- stable under retracts
- 2-out-of-3
- contains eqivs
- stable under cobase change

Lurie calls such a class strongly saturated

Intersections of strongly saturated classes are also strongly saturated, and $\text{Fun}([1], \mathcal{C})$ is also strongly saturated

So for any $T \subseteq \text{Fun}([1], \mathcal{C})$ there is some minimal strongly saturated closure $\bar{T} \supseteq T$

Say S is of small generation if $S = \bar{T}$ for some small T

Thm (Lurie) \mathcal{C} presentable ∞ -cat, $S \subseteq \text{Fun}([1], \mathcal{C})$ is strongly saturated of small generation if $S = S_L$ for some accessible localization $L: \mathcal{C} \rightarrow \mathcal{C}$

Relation to model categories

def A model category is combinatorial if it is presentable and cofibrantly generated
cofibrations are generated by a set

Thm An ∞ -cat \mathcal{C} is presentable if and only if $\mathcal{C} \simeq N_{\Delta}(M_{cf})$ for M a combinatorial model category.