Micah

Homotopy Hypothesis: there should be a theory of "n-categories" where the n-groupoids are models of "n-types"

n-type: a space X where $\pi_k X = 0$ for $k > n$

Ex if X is a 1-type, then ΠX is a 1-groupoid
fundamental groupoid

Taking $n = \infty$, ∞-groupoids should be homotopy types

If we want to study $(\infty, 1)$-categories, this suggests that we should be able to model these by categories enriched in "spaces" (homotopy types)

If we model ∞-cats by quasi-categories, then this "idea" holds

(Jānis + Micah fight)

Ex Top category of compactly generated weak Hausdorff spaces

$\mathrm{Map}_{\mathrm{Top}}(X, Y)$ as the set of continuous maps with the compact-open topology

$$\mathrm{Top} : X \underset{\mathrm{Sing}(-)}{\overset{|-|}{\rightleftarrows}} X_\bullet : s\,\mathrm{Set}$$

is a Quillen adjunction.

So, categories enriched in "spaces" should be the same as categories enriched in sSet

$$\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \longrightarrow \mathrm{Map}(X,Z)$$

$$|\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z)| \longrightarrow |\mathrm{Map}(X,Z)|$$

$$\cong$$

$$|\mathrm{Map}(X,Y)| \times |\mathrm{Map}(X,Z)| \nearrow$$

Cat_Δ is the category of simplicially-enriched categories, $\mathrm{Cat}_{\mathrm{Top}}$ is topologically-enriched categories

So geometric realization induces a functor $\mathrm{Cat}_\Delta \longrightarrow \mathrm{Cat}_{\mathrm{Top}}$

$\mathrm{Sing}(-)$ also commutes with limits, so the same property holds

$\mathcal{C} \in \mathrm{Cat}_\Delta$, $\quad N(\mathcal{C})_0 = $ objects

$$N(\mathcal{C})_1 = \mathrm{Hom}_{\mathcal{C}}(-,-)$$

a 2-simplex in $N(\mathcal{C})$ is a diagram
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

but in this case $h = g \circ f$; but we just want $h \simeq g \circ f$

<u>Recall</u> $N(\mathcal{C})_2 := \mathrm{Hom}_{Cat}([2], \mathcal{C})$ (i.e. $\mathrm{Fun}([2], \mathcal{C})$)

$$[2] \qquad \overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$$

$$\varphi_{02} = \varphi_{12} \circ \varphi_{01}$$

<u>Idea:</u> freely add elements to hom sets of $[n]$ s.t. $\varphi_{ik} \neq \varphi_{jk} \circ \varphi_{ij}$

$$S[2] \qquad \begin{tikzcd} \overset{0}{\bullet} \arrow[r, "\varphi_{01}"] \arrow[rr, bend right, "\varphi_{02}"'] & \overset{1}{\bullet} \arrow[r, "\varphi_{12}"] & \overset{2}{\bullet} \end{tikzcd} \qquad \varphi_{02} \neq \varphi_{01} \circ \varphi_{12}$$

Define $S[n]$ to be this "thickened up" version of $[n]$

So now a functor $S[2] \longrightarrow \mathcal{C}$ is the data of

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

and of a homotopy $h \simeq g \circ f$

This is a functor $\Delta \to Cat_\Delta$ which sends $[n]$ to $S[n]$.

So define a simplicial set $N(\mathcal{C})$

$$\mathrm{Hom}_{sSet}([n], N(\mathcal{C})) \cong \mathrm{Hom}_{Cat_\Delta}(S[n], \mathcal{C})$$

<u>Rmk</u> $sSet := \mathrm{Fun}(\Delta^{op}, Set)$

the Yoneda embedding realizes this as the "colimit completion" of Δ

So functors out of Δ into a category $\mathcal{C}$ with colimits are equivalent to colimit-preserving functors $sSet \to \mathcal{C}$.

So, our functor $\Delta \to Cat_\Delta$ lifts to a colimit-preserving functor $sSet \to Cat_\Delta$

$\implies$ Adjoint functor thm implies that there exists a functor $Cat_\Delta \to sSet$ right adjoint to the first

$$S : sSet \rightleftarrows Cat_\Delta : N$$

<u>To recap:</u> To any simplicial cat $\mathcal{C} \in Cat_\Delta$, we obtain a sSet, $N(\mathcal{C})$

<u>Thm</u> $N(\mathcal{C})$ is an ∞-cat

<u>Thm</u> these functors N and S are equivalences

def The topological nerve

$$\mathrm{Cat}_{\mathrm{Top}} \longrightarrow \mathrm{sSet}$$

is defined by $N \circ \mathrm{Sing}(-)$

Ex Top is now an ∞-cat by associating it with its topological nerve

Mapping Spaces in ∞-categories

Let $\mathcal{C}$ be an ∞-cat
For two objects $x, y \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(X, Y) := \mathrm{Map}_{hS}(X, Y)$$

i.e. $S(\mathcal{C})(X, Y) \in \mathcal{K} \longrightarrow$ simplicial sets localized at weak equivalences

No obvious maps

$$\mathrm{Map}_{hS}(X, Y) \times \mathrm{Map}_{hS}(Y, Z) \longrightarrow \mathrm{Map}_{hS}(X, Z)$$

Jānis

$\mathfrak{S} : \Delta \to Cat_\Delta$ extends to $\mathfrak{S} : sSet \to Cat_\Delta$ with right adjoint $N : Cat_\Delta \to sSet$

$Ho : sSet \rightleftarrows Cat_1 : N$ (not simplicial nerve)

See HTT 1.2.3, Cisinski 1.4–1.6 "Boardman-Vogt constr."

Let $\mathcal{C}_\bullet$ be an ∞-cat.

Goal: Construct a 1-cat $Ho(\mathcal{C})$.

Construct $\pi(\mathcal{C}) \cong Ho(\mathcal{C})$

What is a 1-cat?

- objects
- morphisms
- composition
- identity morphisms
- associativity of comp.

$Ob(\pi(\mathcal{C})) = \mathcal{C}_0$

Morphisms: For every $(\varphi : \Delta^1 \to \mathcal{C}) \in \mathcal{C}_1$, put $\bar{\varphi} \in Edge(\varphi(0), \varphi(1))$

We also have $id_X \in Edge(X, X)$ (degen. 1-simplex)

def let $\bar{\varphi}, \bar{\psi} \in \mathrm{Edge}(X,Y)$. Then $\bar{\varphi}$ and $\bar{\psi}$ are homotopic if $\exists$ a 2-cell $\sigma \in \mathcal{C}_2$ s.t. $\partial_2(\sigma) = \bar{\varphi}$
$\partial_1(\sigma) = \bar{\psi}$
$\partial_0(\sigma) = \overline{\mathrm{id}_Y}$

$$\begin{tikzcd} & Y \arrow[dr, "\overline{\mathrm{id}_Y}"] & \\ X \arrow[ur, "\bar{\varphi}"] \arrow[rr, "\bar{\psi}"'] \arrow[rr, phantom, shift left=3, "\sigma"] & & Y \end{tikzcd}$$

Prop For every $\varphi \in \mathcal{C}_1$ with $\varphi(0) = X$ and $\varphi(1) = Y$, homotopy is an equivalence relation on $\mathrm{Edge}(X,Y)$.

Pf Reflexivity:

$$\begin{tikzcd} & Y \arrow[dr, "\overline{\mathrm{id}_Y}"] & \\ X \arrow[ur, "\bar{\varphi}"] \arrow[rr, "\bar{\varphi}"'] \arrow[rr, phantom, shift left=3, "s_1(\bar{\varphi})"] & & Y \end{tikzcd} \in \mathcal{C}_2$$

Symmetry/trans:

$\bar{\varphi}, \bar{\psi}, \bar{\chi} \in \mathrm{Edge}(X,Y)$

assume $\bar{\varphi} \sim \bar{\psi}$
$\bar{\psi} \sim \bar{\chi}$

$$\begin{tikzcd} & Y \arrow[d, "\overline{\mathrm{id}_Y}"] \arrow[dd, bend left=60, "\overline{\mathrm{id}_Y}"] \\ X \arrow[ur, "\bar{\varphi}"] \arrow[r, "\bar{\psi}"'] \arrow[dr, "\bar{\chi}"'] & Y \arrow[d, "\overline{\mathrm{id}_Y}"] \\ & Y \end{tikzcd}$$

this is $\Lambda^3_2 \to \mathcal{C}$, lifts to $\tau : \Delta^3 \to \mathcal{C}$

Then $d_2(\tau)$ shows that $\bar{\varphi} \sim \bar{\chi}$

for symmetry, take same diagram going other direction.

__def__ for $x, y \in Ob(\pi(\mathcal{C})) = \mathcal{C}_0$, let

$$\mathrm{Hom}_{\pi(\mathcal{C})}(x,y) = \mathrm{Edge}(x,y)/\text{homotopy}$$

$$(\varphi : \Delta^1 \longrightarrow \mathcal{C}) \in \mathcal{C}_1 \rightsquigarrow \bar{\varphi} \in \mathrm{Edge}(x,y) \rightsquigarrow [\bar{\varphi}] \in \mathrm{Hom}_{\pi(\mathcal{C})}(\varphi(0), \varphi(1))$$

__Composition__: $[\bar{\varphi}] \in \mathrm{Hom}_{\pi(\mathcal{C})}(X, Y)$, $\bar{\psi} \in \mathrm{Hom}_{\pi(\mathcal{C})}(Y, Z)$

$$\begin{tikzcd} & Y \arrow[dr, "\bar{\psi}"] & \\ X \arrow[ur, "\bar{\varphi}"] & & Z \end{tikzcd} \quad \text{extends to } \sigma : \Delta^2 \longrightarrow \mathcal{C}$$

$$\Lambda^2_1 \longrightarrow \mathcal{C}$$

$$[\bar{\psi}] \circ [\bar{\varphi}] := [\overline{d_1 \sigma}]$$

__Is comp. well defined?__

① Choice of σ is ok:

$$\begin{tikzcd} & & Z \arrow[d, "id_Z"] \\ X \arrow[r, "\bar{\varphi}"] \arrow[urr, bend left, "\sigma"] \arrow[drr, bend right, "\sigma'"'] & Y \arrow[ur, "\bar{\psi}"] \arrow[dr, "\bar{\psi}"'] \arrow[r, phantom, "s_1 \psi"] & {} \\ & & Z \end{tikzcd}$$

$$\Lambda^3_1 \longrightarrow \mathcal{C}$$

$$\tau : \Delta^3 \longrightarrow \mathcal{C}$$

$$d_2 d_1 \tau = d_1 \sigma$$

$$d_1 d_1 \tau = d_1 \sigma'$$

$$d_0 d_1 \tau = id_Z$$

Now $d_1 \tau$ shows that $\overline{d_1 \sigma} \sim \overline{d_1 \sigma'}$

② Choice of $\bar{\varphi}$ is ok:

$$\begin{tikzcd} & Y \arrow[dr, "\bar{\psi}"] & \\ X \arrow[ur, "\bar{\varphi}'"] \arrow[r, "\bar{\varphi}"] \arrow[rr, bend right] & Y \arrow[u, "\overline{id}_Y"] \arrow[r, "\bar{\psi}"] \arrow[d, phantom, "\sigma"] \arrow[ur, phantom, "s_0\bar{\psi}"] & Z \\ & & \end{tikzcd}$$

$\Lambda^3_1 \longrightarrow \mathcal{C}$

$\tau : \Delta^3 \longrightarrow \mathcal{C}$

So $d_1\tau$ says: $[\bar{\psi}]\circ[\bar{\varphi}'] = [d_1 d_1 \tau]$

$= [d_1\sigma]$

$= [\bar{\psi}]\circ[\bar{\varphi}]$.

③ Choice of $\bar{\psi}$ is ok:

$$\begin{tikzcd} & & Z \\ X \arrow[r, "\bar{\varphi}"] \arrow[rr, bend right, "\sigma"] \arrow[urr, bend right=80, "s_1 d_1 \sigma"'] & Y \arrow[ur, "\bar{\psi}'"] \arrow[r, "\bar{\psi}"] & Z \arrow[u, "\overline{id}_Z"'] \end{tikzcd}$$

$\Lambda^3_2 \longrightarrow \mathcal{C} \quad \Rightarrow \quad \tau : \Delta^3 \longrightarrow \mathcal{C}$

Here $d_2\tau$ shows that

$$[\bar{\psi}']\circ[\bar{\varphi}] = [\bar{\psi}]\circ[\bar{\varphi}].$$

__Identity:__

$$\begin{tikzcd} & Y \arrow[d, "\overline{id_Y}"] \\ X \arrow[ur, "\overline{\varphi}"] \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

$\overline{s_1 \varphi}$ says $[\overline{id_Y}] \circ [\overline{\varphi}] = [\overline{\varphi}]$

$$\begin{tikzcd} & X \arrow[d, "\overline{\varphi}"] \\ X \arrow[ur, "\overline{id_X}"] \arrow[r, "\overline{\varphi}"'] & Y \end{tikzcd}$$

$\overline{s_0 \varphi}$ says $[\overline{\varphi}] \circ [\overline{id_X}] = [\overline{\varphi}]$.

__Associativity:__

$$\begin{tikzcd} X \arrow[r, "\overline{\varphi}"'] \arrow[rr, bend left, ""] \arrow[rrr, bend left=50, ""] & Y \arrow[r, "\overline{\psi}"'] \arrow[rr, bend right, ""] \arrow[r, phantom, bend left, "\sigma"] & Z \arrow[r, "\overline{\chi}"'] \arrow[r, phantom, bend left=40, "\sigma''"] \arrow[l, phantom, bend right=20, "\sigma'"'] & W \end{tikzcd}$$

$$\Lambda^3_2 \longrightarrow \mathcal{C} \Rightarrow \tau : \Delta^3 \longrightarrow \mathcal{C}$$

$$\begin{aligned} [\overline{\chi}] \circ ([\overline{\psi}] \circ [\overline{\varphi}]) &= [\overline{d_1 \sigma''}] \\ &= [\overline{d_1 d_2 \tau}] \\ &= [\overline{d_1 \sigma'}] \circ [\overline{\varphi'}] \\ &= ([\overline{\chi}] \circ [\overline{\psi}]) \circ [\overline{\varphi}] \end{aligned}$$

Therefore, $\pi(\mathcal{C})$ is a 1-cat. $\square$

Ex

- View a $\mathcal{C} \in Cat_1$ as $\mathcal{C} \in Cat_\infty$. Then $Ho(\mathcal{C}) = \mathcal{C}$, since $\mathcal{C}_{\geq 2} = \mathcal{C}_1$.

- Let X be a top. space. Then $Ho(Sing(X)) = \Pi_1 X$

- Let $\mathcal{C} \in Cat_\Delta$, then $N(\mathcal{C}) \in Cat_\infty$. If $Map_{\mathcal{C}}(X, Y)$ is a Kan complex, then $Ho(N(\mathcal{C})) = h\mathcal{C}$

Maximilien

limits and colimits in ∞-categories

Goal: introduce the notion formally then present an equivalent approach with homotopy (co)limits via topological (or simplicial) enrichments

def Let $\mathcal{C}$ be an ∞-category. Let $p : K \longrightarrow \mathcal{C}$ be a map of simplicial sets. A limit of p is a final object in $\mathcal{C}_{/p}$. A colimit of p is an initial object in $\mathcal{C}_{p/}$

① Quick reminder for ordinary categories

$\mathcal{C}$ an ordinary cat, $X \in Ob(\mathcal{C})$.

over category $\mathcal{C}_{/X}$: Ob: $\begin{array}{c} X' \\ \downarrow \\ X \end{array} \in \mathcal{C}$

Mor: $\begin{tikzcd} X' \arrow[d] \\ X \end{tikzcd} \quad \begin{tikzcd} X'' \arrow[d] \\ X \end{tikzcd} = \begin{tikzcd} X' \arrow[rr] \arrow[dr] & & X'' \arrow[dl] \\ & X & \end{tikzcd}$

Can be generalized: $X : [0] \longrightarrow \mathcal{C}$
↳ cat w/ 1 obj + id

Given a functor $p : I \longrightarrow \mathcal{C}$, denote by $\mathcal{C}_{/p}$ the slice category of cones of p.

$\mathcal{C}_{/p}$ are objects of the form

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ p(i) \arrow[rr] & & p(j) \end{tikzcd}$$

Dually we have notion of cocones $\mathcal{C}_{p/}$.

Recall: $\lim(p)$ = terminal object of $\mathcal{C}_{p/}$.

② Joins of ∞-cats

Given A, B ordinary cats, define

$A \star B$: $\quad \mathrm{ob}(A \star B) = \mathrm{ob}(A) \amalg \mathrm{ob}(B)$

$$\mathrm{Hom}_{A \star B}(X,Y) \begin{cases} \mathrm{Hom}_A(X,Y), & X, Y \in A \\ \mathrm{Hom}_B(X,Y), & X, Y \in B \\ *, & X \in A,\ Y \in B \\ \emptyset, & X \in B,\ Y \in A. \end{cases}$$

def $[0] \star B =: B^{\triangleleft}$, the category of cones of B

$B \star [0] =: B^{\triangleright}$, the category of cocones of B

ex

$$\begin{tikzcd} & \bullet \arrow[d] \\ B = \quad \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$B^{\triangleleft} \simeq \begin{tikzcd} -\infty \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

This construction can be extended to simplicial sets such that

$$N(A) * N(B) \xrightarrow{\cong} N(A * B)$$

(regular nerve)

def $K, L \in sSet$. $(K * L)_m = K_m \cup L_m \cup \bigcup_{i+j=m-1} K_i \times L_j$

ex $\Delta^0 * \Delta^0 \cong \Delta^1$

In general, $\Delta^n = \underbrace{\Delta^0 * \cdots * \Delta^0}_{n+1 \text{ times}}$

Even more general, $\Delta^i * \Delta^j \cong \Delta^{i+1+j}$

$\exists$ Canonical inclusions $K \hookrightarrow K * L$, $L \hookrightarrow K * L$

$\rightsquigarrow$ $\forall L \in sSet$, $sSet \longrightarrow (sSet)_{L/}$

$K \longmapsto K * L$,

this functor preserves colimits

$\underline{\text{def}}$ $\Delta^0 \star L =: L^{\triangleleft}$ cone of L

$K \star \Delta^0 =: K^{\triangleright}$ cocone of K

$\underline{\text{ex}}$
$$\Lambda^2_2 = \begin{tikzcd} & c_1 \arrow[d] \\ c_0 \arrow[r] & c_2 \end{tikzcd}$$

$$(\Lambda^2_2)^{\triangleleft} \cong \Delta^1 \times \Delta^1$$

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] \arrow[dr] & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd}$$

③ Sliced ∞-categories

In ordinary categories, given a functor $p: A \to B$, notice $B_{p/}$ is characterized by

$$\mathrm{Fun}(\mathcal{C}, B_{p/}) \cong \mathrm{Fun}_p(\mathcal{C} \star A, B)$$

$$\begin{tikzcd} \mathcal{C} \star A \arrow[r] & B \\ A \arrow[u] \arrow[ur, "p"'] & \end{tikzcd}$$

$\mathcal{C} = [0]$ $A = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$

$$B_{p/} = \begin{tikzcd} x \arrow[r] \arrow[d] & p(\cdot) \arrow[d] \\ p(\cdot) \arrow[r] & p(\cdot) \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} -\infty \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \xrightarrow{\text{functor}} B$$

Worthwhile rewriting the iso:

$$\mathrm{Mor}_{Cat}(\mathcal{C}, B_{/p}) \cong \mathrm{Hom}_{Cat_{A/}}\left(\begin{array}{c} A \\ \downarrow \\ C \star A \end{array}, \begin{array}{c} A \\ \downarrow p \\ B \end{array}\right)$$

<u>Prop</u> Let $p : L \to \mathcal{C}$ a map of sSet where $\mathcal{C}$ is an ∞-cat. Then there is an ∞-cat $\mathcal{C}_{p/}$ characterized by

$\forall K \in sSet$: $\quad \mathrm{Hom}_{sSet}(K, \mathcal{C}_{p/}) \cong \mathrm{Hom}_{sSet_{L/}}\left(\begin{array}{c} L \\ \downarrow \\ K \star L \end{array}, \begin{array}{c} L \\ \downarrow p \\ \mathcal{C} \end{array}\right)$

$$(\mathcal{C}_{/p})_n = \mathrm{Hom}_{sSet_{L/}}\left(\begin{array}{c} L \\ \downarrow \\ \Delta^n \star L \end{array}, \begin{array}{c} L \\ \downarrow p \\ \mathcal{C} \end{array}\right)$$

(Dually, we have $\mathcal{C}_{/p}$ where replace $K \star L$ with $L \star K$)

④ initial objects / final objects

An obj $X \in \mathcal{C}$ is <u>final</u> if in its homotopy $Ho(\mathcal{C})$, : $\forall Y \in \mathcal{C}$, $\mathrm{Mor}_{Ho(\mathcal{C})}(X, Y) \sim *$

weak

$\mathcal{C}$ can be top cat, simplicial cat, or ∞-cat.

5 ∞-(co)limits as homotopy (co)-limits

Unpacking the definition: $\lim(K \xrightarrow{p} \mathcal{C})$ is the final object of $\mathcal{C}_{/p}$. An object of $\mathcal{C}_{p/}$ is a vertex of $(\mathcal{C}_{p/})_0$.

Recall: $(\mathcal{C}_{/p})_0 = \mathrm{Hom}_{sSet_{K/}}\left(\begin{array}{c} K \\ \downarrow \\ \Delta^n * K \end{array}, \begin{array}{c} K \\ \downarrow p \\ \mathcal{C} \end{array} \right)$

i.e. a functor $\bar{p} : K^{\triangleleft} = \Delta^0 * K \longrightarrow \mathcal{C}$ such that

$$\bar{p}|_K = p$$

Often we denote $\bar{p}(-\infty) \in \mathcal{C}$ as $\lim(p)$

$\rightsquigarrow$ the usual nerve preserves (co)limits

<u>Recall</u> Given an ∞-cat $\mathcal{C}$, it is enriched topologically (or simplicially):

given objects $X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X,Y)$

can be regarded as the object in the homotopy category of spaces representing the space of maps from X to Y in $Ho(\mathcal{C})$

Recall 2: For usual limits in ordinary cats:

let $L = \lim(I \xrightarrow{p} \mathcal{C})$, then

$$\forall X \in \mathcal{C}: \quad \mathrm{Map}_{\mathcal{C}}(X, L) \xrightarrow{\simeq} \mathrm{Mor}_{\mathcal{C}^I}(\underline{X}, p)$$

const. diag.

Prop Let $\mathcal{C}$ be an ∞-cat and K a sSet, $p: K \to \mathcal{C}$ a sSet map. Then

$\bar{p}: K^{\triangleleft} \longrightarrow \mathcal{C}$ is a limit of $p = \bar{p}|_K$

$\Longleftrightarrow$ if we denote $L = \bar{p}(-\infty)$, $\forall X \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(X, L) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}, p)$$

Ex Recall that homotopy product in Top is given by the product in Ho(Top)

If $\mathcal{C}$ is a topological cat, then homotopy product $\prod_\alpha Y_\alpha \in \mathcal{C}$ is determined

$$\forall X \in \mathcal{C}: \quad \mathrm{Map}(X; \prod Y_\alpha) \xrightarrow{\sim} \prod_\alpha \mathrm{Map}(X, Y_\alpha)$$

weak

ex recall that the homotopy pullback in Top:

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is given by the space

$$\{(x,y,h) \in X \times Y \times \mathrm{Map}([0,1],Z) \mid h(0)=f(x),\ h(1)=g(y)\}$$
$$\subseteq X \times Y \times \mathrm{Map}([0,1],Z)$$

For a topological cat $\mathcal{C}$, P is a htpy pullback of $X \to Z$ in $\mathcal{C}$ if:

$\forall W$: $\mathrm{Map}_{\mathcal{C}}(W,P) \xrightarrow{\sim} \mathrm{Map}(W,X) \times^{h}_{\mathrm{Map}(W,Z)} \mathrm{Map}(W,Y)$

def Let $\mathcal{C}$ be a topological category. Define the topological nerve of $\mathcal{C}$:

$$N_{Top}(\mathcal{C}) = N_{\Delta}(\mathrm{Sing}(\mathcal{C}))$$

def Let $\mathcal{S}$ be the ∞-cat of spaces defined as $N_{\Delta}(\mathrm{Kan})$ or $N_{Top}(\mathrm{CW})$

Thm Let $\mathcal{I}, \mathcal{C}$ be topological categories. Let $p: \mathcal{I} \to \mathcal{C}$ be a continuous functor. Suppose we have a cone $(C, \{\eta_i : C \to p(i)\}_{i \in I})$. $(C, \{\eta_i\})$ is a homotopy limit of p $\iff$

$$p = N_{Top}(p) : N_{Top}(\mathcal{I}) \to N_{Top}(\mathcal{C})$$

can be extended to a functor

$$\bar{p} : N_{Top}(\mathcal{I})^{\triangleleft} \longrightarrow N_{Top}(\mathcal{C})$$

is an ∞-limit in $N_{Top}(\mathcal{C})$.

∞-limit in $\mathcal{S}$ = htpy limits in spaces

$\uparrow$ topological enrichments

∞-limits in any ∞-cat

$\exists$ statement for simplicial as well

Harry

Presentable (ordinary) categories

- bridge the gap between small categories and proper classes
- presentable categories are "generated" by small categories

Ex Ab is a proper class, but Ab^{fg} is small

Recall for any small A, we have cocomplete presheaf category $Fun(A^{op}, Set)$

$$\begin{tikzcd} A \arrow[r, "\text{yoneda}"] \arrow[d] & Fun(A^{op}, Set) \arrow[dl, dashed] \\ B & \end{tikzcd}$$

$$x \longmapsto Hom(-, x)$$

B cocomplete

think of this as <u>cocompletion</u> or "free generation"

Thm For A small and $\mathcal{C}$ cocomplete, the restriction to A along the Yoneda embedding gives an equivalence

$$Fun^L(Fun(A^{op}, Set), \mathcal{C}) \xrightarrow{\sim} Fun(A, \mathcal{C})$$

colim preserving functors

Cor $A = *$, $\mathrm{Fun}^L(\mathrm{Set}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}$

Thm (Adjoint Functor Thm) (Freyd)

For $F : \mathcal{C} \to \mathcal{D}$, $\mathcal{C}, \mathcal{D}$ cocomplete, F is a left adjoint if and only if F preserves colimits and the "solution set condition" is satisfied

SSC: (roughly) some class of morphisms is a set

Presentable categories form a suitably large class of categories for which we can drop SSC.

def Let κ be a regular cardinal. Then a κ-filtered colimit is given by a filtered system where objects + morphisms are less than κ.

A category $\mathcal{C}$ is κ-accessible if it contains some small subcat $\mathcal{D} \subseteq \mathcal{C}$ such that:

- every $X \in \mathcal{C}$ is a κ-filtered colim of objects in $\mathcal{D}$ ($\mathcal{D} \to \mathcal{C}$ is dense)
- for $d \in \mathcal{D}$, $\mathrm{Hom}(d, -)$ commutes with κ-filtered colimits (d's are κ-compact in $\mathcal{C}$)

def $\mathcal{C}$ is <u>accessible</u> if it is κ-accessible for some κ and $F: \mathcal{C} \to \mathcal{D}$ is accessible if it preserves filtered colimits

def A category $\mathcal{C}$ is <u>presentable</u> if it is accessible + cocomplete

<u>Note</u>: Any presentable cat is also complete

<u>Ex</u>
- Set, as generated by $*$
- Any presheaf category on a small category $\mathrm{Fun}(A^{op}, \mathrm{Set})$
- $\mathrm{sSet} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$
- Mod_R generated by f.g. projectives
- $\mathrm{Ch}(R)$ generated by perfect complexes
- Quasicoherent $\mathcal{O}_X$-modules on any scheme

<u>Classification for presentable categories</u>

A category $\mathcal{C}$ is presentable if and only if it is a localization of a presheaf category) on a small category

accessible

$$A \xrightarrow{\text{Yoneda}} \mathrm{Fun}(A^{op}, \mathrm{Set}) \xrightarrow{\text{localization}} \mathcal{C}$$

free generation $\qquad\qquad\qquad$ relations

<u>Adjoint Functor Thm for Presentable Categories:</u>

$F : \mathcal{C} \to \mathcal{D}$, $\mathcal{C}, \mathcal{D}$ presentable

1. F is a left adjoint if and only if it preserves colimits
2. F is a right adjoint if and only if it preserves limits and is accessible.

<u>Presentable ∞-categories</u>

<u>Recall</u> ∞-category of spaces $\mathcal{S} = N_\Delta(\mathrm{Kan})$

for some small simplicial set K, define ∞-cat of presheaves

$$\mathcal{P}(K) := \mathrm{Fun}(K^{op}, \mathcal{S})$$

<u>Recall</u> We have an adjunction

$$(\mathfrak{C}[-], N_\Delta) : s\mathrm{Set} \rightleftarrows \mathrm{Cat}_\Delta$$

simplicial thickening $\qquad$ coherent nerve

For some simplicial set, take $C[K]$ and $C[K^{op}]$

$$C[K] \times C[K^{op}] \longrightarrow Kan$$
$$(X, Y) \longmapsto Sing|Hom_{C[K]}(X, Y)|$$

We compose

$$C[K \times K^{op}] \longrightarrow C[K] \times C[K^{op}] \longrightarrow Kan$$

pass to adjoint

$$K \times K^{op} \longrightarrow N_{\Delta}(Kan) \simeq S$$

exponential law

$$K \longrightarrow Fun(K^{op}, S)$$ this is ∞-yoneda embedding

- Fully faithful and satisfies necessary universal property

Thm $Fun^L(\mathcal{P}(\mathcal{D}), \mathcal{C}) \xrightarrow{\sim} Fun(\mathcal{D}, \mathcal{C})$

Cor Take $\mathcal{D} = \Delta^0$, get $Fun^L(S, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}$

"The ∞-cat of spaces is freely generated by the zero simplex Δ^0"

def An ∞-cat is <u>presentable</u> if it is cocomplete and accessible.

A functor between ∞-cats is a <u>localization</u> if it has a fully faithful right adjoint

Classification of presentable ∞-cats

∞-cat $\mathcal{C}$ is presentable if and only if it is an accessible localization of $\mathcal{P}(\mathcal{D})$ for some small ∞-cat $\mathcal{D}$.

Towards understanding accessible localizations of ∞-categories:

$(L, R) : \mathcal{C} \rightleftarrows \mathcal{D}$, denote $L : \mathcal{C} \to \mathcal{D} \to \mathcal{C}$ as well

let S_L = the class of maps in $\mathcal{C}$ sent to equivalences by L

Localizations are completely determined by S_L

In this case,

$S_L \subseteq \mathrm{Fun}([1], \mathcal{C})$:

- closed under colimits
- stable under retracts
- 2-out of-3
- contains equivs
- stable under cobase change

Lurie calls such a class <u>strongly saturated</u>

Intersections of strongly saturated classes are also strongly saturated, and $\mathrm{Fun}([1], \mathcal{C})$ is also strongly saturated

So for any $T \subseteq \mathrm{Fun}([1], \mathcal{C})$ there is some minimal strongly saturated closure $\overline{T} \supseteq T$

Say S is of small generation if $S = \overline{T}$ for some small T

<u>Thm</u> (Lurie) $\mathcal{C}$ presentable ∞-cat, $S \subseteq \mathrm{Fun}([1], \mathcal{C})$ is strongly saturated of small generation if $S = S_L$ for some accessible localization $L : \mathcal{C} \longrightarrow \mathcal{C}$

<u>Relation to model categories</u>

<u>def</u> A model category is <u>combinatorial</u> if it is presentable and $\underbrace{\text{cofibrantly generated}}_{\text{cofibrations are generated by a set}}$

<u>Thm</u> An ∞-cat $\mathcal{C}$ is presentable if and only

$$\mathcal{C} \simeq N_\Delta(M_{cf})$$ for M a combinatorial model category.